SUPERPOSITION PRINCIPLE: PHASE VELOCITY AND GROUP VELOCITY.

The principle of superposition of waves states that if two or more mutually independent waves travel through the same region of space, the total wave field parameter is the vector sum of the individual wave field parameters.

Let $\overrightarrow{\psi_1}$, $\overrightarrow{\psi_2}$, $\overrightarrow{\psi_3}$ ....$\overrightarrow{\psi_n}$ be n number os waves travelling through the same region of space, then accorfing to the principle of superposition of waves, the resultant wave is given by:

$$\overrightarrow{\psi_R} = \overrightarrow{\psi_1} + \overrightarrow{\psi_2} + \overrightarrow{\psi_3} + \ldots\ldots\ldots + \overrightarrow{\psi_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5.1)$$

This principle can be applied to many branches of Physics such as optics, acoustics etc. It allows us to analyse any complicated wave motion as a combination of a number of simple harmonic waves. Since according to de Broglie, a wave motion is assigned to a moving particle, the principle of superposition of waves can be extended to matter waves also.

The superposition principle in quantum mechanics is basically different from that in classical physics. Superposition of two classical waves '$\overrightarrow{\psi_1}$' and '$\overrightarrow{\psi_2}$' gives rise to a new wave of wave field parameter ψ_R given by : $\psi_R = \psi_1 + \psi_2$, with different amplitude and phase as compared to the component waves But two quantum states on superposition does not lead to a new state. $\psi^*\psi$, which is actually the amplitude square of different superposed states , determine the probability of the respective states, when measurements are made repeatedly on the same system. The process of measurement throws the system in that state which occurs more frequently than others if its probability exceeds those of others.

Concept of phase velocity:

Let a plane simple harmonic wave moving in positive X direction be given by:

$$\psi = a\mathrm{Sin}(\omega t - kx) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.1) \qquad \text{where } \omega = 2\pi\nu \text{ and } k = \frac{2\pi}{\lambda},$$

a is the amplitude, ν is the frequency and λ is the wavelength.

From equation (5.1) the phase of the wave at (x,t) is given by :

$$\phi(x,t) = \omega t - kx \Rightarrow \quad \frac{\partial\phi}{\partial t} = \omega - k\frac{dx}{dt} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.2)$$

For a point of constant phase $\frac{\partial\phi}{\partial t} = 0$, which gives: $\qquad \omega - k\frac{dx}{dt} = 0$

or $\omega = k\frac{dx}{dt} = ku$ ……………. …………………………(5.3)

where $u = \frac{dx}{dt}$ = the velocity with which the wave propagates in X direction This velocity is called the **phase velocity of the wave.** The phase velocity is also called **wave velocity, of a monochromatic wave,** i.e. the velocity with which a definite phase of the wave propagates through a medium**.** It is denoted by v_g. From equation (5.3) the expression for phase velocity is given by:

$$\boxed{u = v_p = \frac{\omega}{k}} \quad \text{……………………………………………(5.4)}$$

Concept of group velocity

For a monochromatic wave the successive waves are similar and hence indistinguishable. So it is not possible to measure the phase velocity experimentally. The way out is to associate a distinguishing mark to the infinitely extended smooth wave.

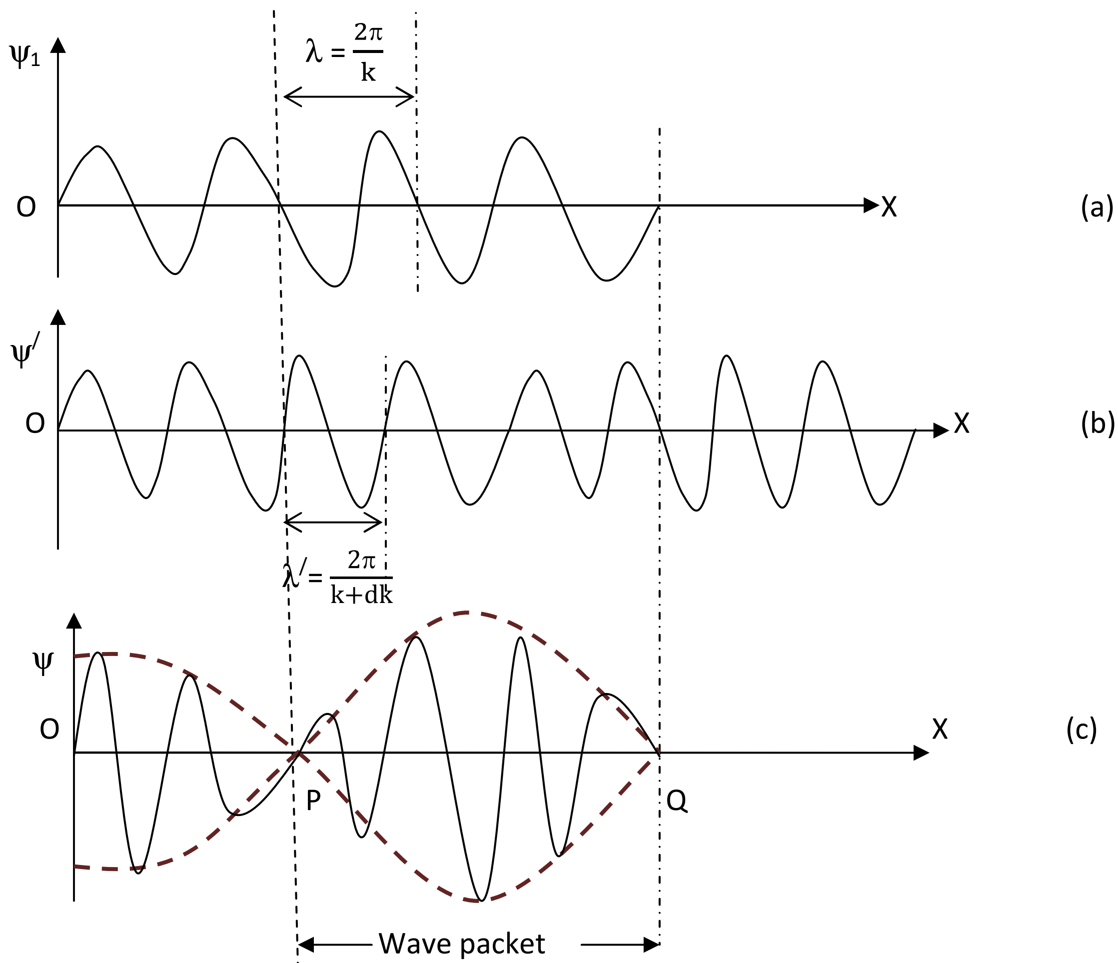

Fig. 5.1. Demonstration of phase velocity and group velocity.

The only way of doing this is to superpose other wave trains of slightly different wavelengths resulting in the formation of a hump at definite places on the smooth wave by mutual interference of wave trains (fig. 5.1). The hump or the envelope of waves is called a wave packet or wave group.

The velocity with which the wave packet travels in a dispersive medium is called the group velocity v_g which is different from that of the component monochromatic wave trains.

Relation between phase velocity and group velocity.

There is a definite relationship between phase velocity and group velocity. To determine the relation, a group of two plane waves ψ_1 and $\psi^/$ with frequencies ν and $\nu^/$ is considered propagating in positive X direction. Accordingly their wavelengths are also different, being 'λ' for one and '$\lambda^/$' for the other. Let v_{p1} be the phase velocity of ψ and $v_p^/$ be the phase velocity of $\psi^/$.

Let the two waves be represented by :

$$\left.\begin{aligned} \psi_1 &= a\mathrm{Sin}(\omega t - kx) \\ \text{And} \quad \psi^/ &= a\mathrm{Sin}(\omega^/ t - k^/ x) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(5.5)$$

For simplicity the amplitude of the two waves are taken to be equal. Due to differences in frequency and wavelength, the value of angular frequency of ψ_1 is ω and that $\psi^/$ is $\omega^/$. The value of wave number is 'k' for ψ_1 and $k^/$ for $\psi^/$. On superposition the resultant of the two waves is given by:

$$\psi = \psi_1 + \psi^/$$

$$= a\mathrm{Sin}(\omega t - kx) + a\mathrm{Sin}(\omega^/ t - k^/ x)$$

$$= 2a\mathrm{Cos}\left\{\frac{1}{2}(k - k^/)x - \frac{1}{2}(\omega - \omega^/)t\right\} \mathrm{Sin}\left\{\frac{1}{2}(k + k^/)x - \frac{1}{2}(\omega + \omega^/)t\right\} \ldots\ldots(5.6)$$

Equation (5.6) denotes that the resultant wave has an angular frequency $\frac{1}{2}(\omega + \omega^/)$ and wave number $\frac{1}{2}(k + k^/)$.

So the sinusoidally varying part of the resultant wave is $= \mathrm{Sin}\left\{\frac{1}{2}(k + k^/)x - \frac{1}{2}(\omega + \omega^/)t\right\}$

Hence the phase velocity of the resultant wave is :

$$V_p = = \frac{\text{angular frequency of the resultant wave}}{\text{wave number of the resultant wave}}$$

$$= \frac{\frac{1}{2}(\omega + \omega^{/})}{\frac{1}{2}(k+k^{/})}$$

$$= \frac{\omega + \omega^{/}}{k+k^{/}}$$

Assuming that the difference between ω and $\omega^{/}$ is very small we can write :

$\omega + \omega^{/} \approx 2\omega$ and $k + k^{/} \approx 2k$. So the expression for phase velocity becomes :

$$\boxed{V_p = \frac{\omega}{k}} \quad \text{...(5.7)}$$

The amplitude of the resultant wave is given by:

$$A = 2a\ \mathrm{Cos}\left\{\frac{1}{2}(k - k^{/})x - \frac{1}{2}(\omega - \omega^{/})t\right\} \quad \text{.....................(5.8)}$$

So the amplitude varies slowly according to the cosine term (refer to the envelope of fig. 5.1) with angular frequency $\frac{1}{2}(\omega - \omega^{/})$ and wave number $\frac{1}{2}(k - k^{/})$. So the amplitude itself has the form of a wave. So it can be said that the envelope or wave packet moves with the group velocity v_g given by:

$$V_g = \frac{\text{angular frequency of the wave packet}}{\text{wave number of the wave packet}}$$

$$= \frac{\frac{1}{2}(\omega - \omega^{/})}{\frac{1}{2}(k-k^{/})}$$

$$= \frac{\omega - \omega^{/}}{k-k^{/}}$$

$$= \frac{d\omega}{dk}$$

$$\boxed{V_g = \frac{d\omega}{dk}} \quad \text{...(5.9)}$$

Now $V_p = \frac{\omega}{k}$

Differentiating with respect to k

$$\frac{dv_p}{dk} = \frac{1}{k}\frac{d\omega}{dk} - \frac{\omega}{k^2} = \frac{i}{k}\left(\frac{d\omega}{dk} - \frac{\omega}{k}\right) = \frac{1}{k}\left(v_g - v_p\right) \quad \text{................(5.10)}$$

So the relation between phase velocity and group velocity is:

$$\boxed{v_g = V_p + k\frac{dv_p}{dk} = V_p - \lambda\frac{dv_p}{d\lambda}} \quad \text{.......................................(5.11)}$$

Where $k = \frac{2\pi}{\lambda}$ = $dk = -\frac{1}{\lambda^2}d\lambda$

For a non dispersive medium, wave velocity i.e. phase velocity is independent of wavelength. Hence $\frac{dv_p}{d\lambda}$ = 0. Hence from equation (5.11) the phase velocity is equal to the group velocity.

Wave packet and its motion

At a given instant the effect of a particle in motion is located over a small region of space Δx so that it can be represented by a wave packet of very small extension. A plane wave travelling along positive X axis may be represented as:

$$\psi(x,t) = Ae^{i(\omega t - kx)} \quad \text{...(5.12)}$$

Where, both the real part and the imaginary part individually represent the same wave

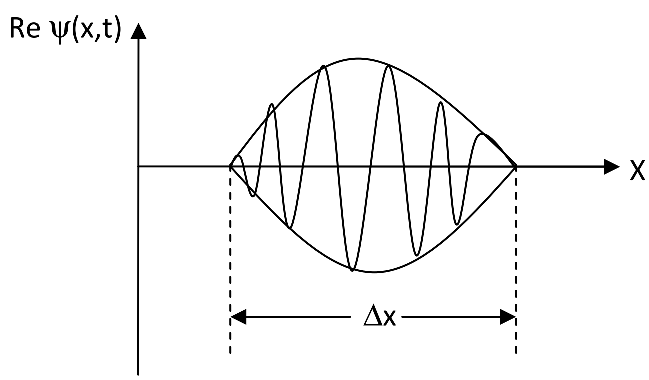

Fig: 5.2. Form of a typical wave packet.

Mathematically a wave packet of small extension Δx can be constructed by superimposing a number of plane waves with slightly varying the 'k' values from an average $k = k_0$ travelling simultaneously along positive X direction. . Fig (2.9) shows a typical form of wave packet, where the real part of $\psi(x,t)$ is plotted as a function of x at a given instant 't'.

By Fourier theorem the wave packet can be expressed by a wave function:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} A(k)e^{i(\omega t - kx)}dk \quad ....(5.13)$$

Where A(k), the amplitude of the harmonic component, is a function of k and is called the Fourier transform of $\psi(x,t)$. Hence A(k) is given by:

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \psi(x,t)e^{-i(\omega t - kx)}dx \quad ....(5.14)$$

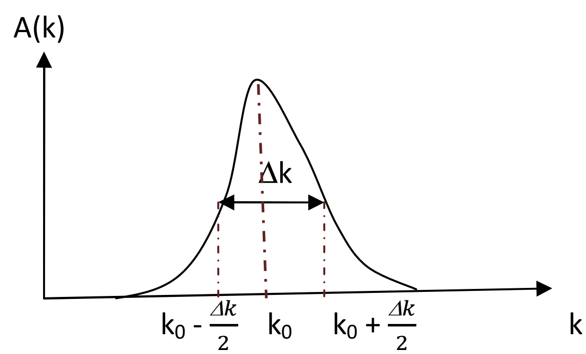

Fig. 5.3. Shape of the function A(k) centred at $k = k_0$

Fig (5.3) shows the shape of A(k) which is considered to be Gaussian. Since A(k) does not depend on 't' hence : $\frac{dA(k)}{dt} = 0$ ....(5.15)

As the particle is localized it is required to consider a wave group which travels without any change of shape. This is possible only if A(k) is zero everywhere except for a small range of k values. As shown in the fig. (5.3) the 'k' values, range within an interval of Δk, from $k_0 - \frac{\Delta k}{2}$ to $k_0 + \frac{\Delta k}{2}$. where $\Delta k \ll k_0$. Hence $\omega(k)$ can be expanded in a Taylor series about k_0 and only the first order term is retained.

$$\omega(k) = \omega(k_0) + (k - k_0)\left(\frac{d\omega}{dk}\right)_{k_0} \quad ....(5.16)$$

Substituting $\omega(k_0) = \omega_0$ and $\left(\frac{d\omega}{dk}\right)_{k_0} = \frac{d\omega}{dk}$ in equation (5.13)

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} A(k)e^{i\left[\left\{\omega_0+(k-k_0)\frac{d\omega}{dk}\right\}t-kx\right]}dk \quad \text{.........................(5.17)}$$

Multiplying and dividing equation (5.17) by e^{ik_0x} we have

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} A(k)e^{i\left[\left\{\omega_0+(k-k_0)\frac{d\omega}{dk}\right\}t-kx\right]}.\,e^{ik_0x}.\,e^{-ik_0x}dk$$

$$= \frac{1}{\sqrt{2\pi}}.\,e^{-ik_0x}\int_{-\infty}^{+\infty} A(k)e^{i\left[\left\{\omega_0+(k-k_0)\frac{d\omega}{dk}\right\}t-kx\right]}.\,e^{ik_0x}dk$$

$$= \frac{1}{\sqrt{2\pi}}.\,e^{-ik_0x}.\,e^{i\omega_0t}\int_{-\infty}^{+\infty} A(k)e^{i\left[\left\{(k-k_0)\frac{d\omega}{dk}\right\}t-kx\right]}.\,e^{ik_0x}dk$$

$$= \frac{1}{\sqrt{2\pi}}.\,e^{i(\omega_0t-k_0x)}\int_{-\infty}^{+\infty} A(k)e^{i\left[\left\{(k-k_0)\frac{d\omega}{dk}\right\}t\right]}.\,e^{-i(k-k_0)x}dk$$

$$= \frac{1}{\sqrt{2\pi}}.\,e^{i(\omega_0t-k_0x)}\int_{-\infty}^{+\infty} A(k)e^{i\left[(k-k_0)\left(\frac{d\omega}{dk}t-x\right)\right]}dk \quad \text{............................((5.18)}$$

Or $\psi(x,t) = F(x)\,.\,e^{i(\omega_0t-k_0x)}$ ....(5.19)

Where $F(x) = \frac{1}{\sqrt{2\pi}}\int_{k_0-\frac{\Delta k}{2}}^{k_0+\frac{\Delta k}{2}} A(k)e^{i\left[(k-k_0)\left(\frac{d\omega}{dk}t-x\right)\right]}dk$ ....(5.19)

if A(k) is assumed to be zero everywhere except for a small range of k values between $k_0 - \frac{\Delta k}{2}$ to $k_0 + \frac{\Delta k}{2}$ so the limits of integration have been changed accordingly.

Equation (5.19) represents a plane wave with wave number k_0 and frequency ω_0 modulated by F(x,t) which depends on x and t through the term $\left(\frac{d\omega}{dk}t - x\right)$ and determines the shape of the envelope of the wave packet. Thus it is evident that the wave packet travels undistortedly with a group velocity:

$$V_g = \frac{\frac{d\omega}{dk}}{1} = \frac{d\omega}{dk} \quad \text{................................(5.20)}$$

While the envelope of the wave packet moves with a velocity v_g, the individual waves of the packet move with the phase velocity v_p. It is actually the wave packet which carries the energy and momentum. The group velocity can be measured experimentally.

To show that the group velocity of a wave associated with a material particle is same as the particle velocity 'v':

Group velocity $V_g = \frac{d\omega}{dk} = \frac{d\omega}{dE}\frac{dE}{dp}\frac{dp}{dk}$ ………………………………(5.21)

where E is the energy and p is the momentum of the particle.

If a wave of frequency ν is associated with the particle then : $E = h\nu = \frac{h\omega}{2\pi}$. or $\omega = \frac{2\pi E}{h}$

So $\frac{d\omega}{dE} = \frac{2\pi}{h}$ ………………………………(5.22)

If λ be the wavelength of the wave associated with the particle then: $p = \frac{h}{\lambda} = \frac{hk}{2\pi}$

So $\frac{dp}{dk} = \frac{h}{2\pi}$ ………………………………(5.23)

Substituting (5.22) and (5.23) in equation (5.21) the expression for group velocity becomes:

$V_g = \frac{dE}{dp}$ ………………………………………(5.24)

Now for a non relativistic free particle travelling with a velocity 'v' and momentum 'p' the energy E is kinetic energy

So $E = \frac{p^2}{2m}$, which gives : $\frac{dE}{dp} = \frac{p}{m} = \frac{mv}{m} = v$……………………(5.25)

This gives : $\boxed{V_g = v}$ ……………………………………(5.26)

For a relativistic free particle : $E^2 = p^2c^2$, which gives $\frac{dE}{dp} = \frac{c^2p}{E} = \frac{c^2mv}{mc^2} = v$

So again $v_g = v$

Thus the group velocity of a wave associated with a material particle is same as the particle velocity 'v'

Phase velocity and group velocity of matter waves:

According to de Broglie hypothesis a free particle of mass 'm' moving with a velocity 'v' has associated with it a wave of frequency 'ν' and wavelength 'λ' given by :

$\nu = \frac{E}{h} = \frac{mc^2}{h}$ ....(5.27), where 'm' is the relativistic mass. Hence substituting $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ in equation (5.27):

$$\boxed{\nu = \frac{m_0 c^2}{h\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0 c^2}{h\sqrt{1-\beta^2}}} \qquad \text{......................................(5.28)}$$

Where $\beta = \frac{v}{c}$, and 'c' is the velocity of light in free space.

$$\boxed{\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h\sqrt{1-\beta^2}}{m_0 v}} \qquad \text{..(5.29)}$$

Phase velocity of de Broglie wave of a particle is:

$$V_p = \nu\lambda = \frac{m_0 c^2}{h\sqrt{1-\beta^2}} \cdot \frac{h\sqrt{1-\beta^2}}{m_0 v}$$

Or $\quad V_p = \frac{c^2}{v}$ ....(5.30)

But it has been shown that the particle velocity is equal to the group velocity of the wave associated with it.

So $\quad V_p = \frac{c^2}{v_g}$

Or $\quad \boxed{V_p . V_g = c^2}$ ....(5.31)

Equation (5.31) gives the relation between phase velocity and group velocity of matter waves.

Using relativistic energy expression we have: $E^2 = p^2c^2 + m_0^2c^4$, where E is the total energy.

Hence $h^2\nu^2 = \frac{h^2}{\lambda^2}c^2 + m_0^2c^4$ ....(5.32)

Substituting $V_p = \nu\lambda$ in (5.32) we have:

$$\frac{h^2 v_p^2}{\lambda^2} = \frac{h^2}{\lambda^2}c^2 + m_0^2 c^4$$

Or $$V_p^{\,2} = c^2 + \frac{m_0^2 c^4 \lambda^2}{h^2}$$

$$= c^2\left(1 + \frac{m_0^2 c^2 \lambda^2}{h^2}\right)$$

Hence $$\boxed{V_P = c\sqrt{\left(1 + \frac{m_0^2 c^2 \lambda^2}{h^2}\right)}} \qquad \text{…………………………(5.33)}$$

Equation (5.33) gives the phase velocity of de Broglie wave. Since $\sqrt{\left(1 + \frac{m_0^2 c^2 \lambda^2}{h^2}\right)} > 1$, so equation (5.33) points out that the phase velocity is greater than 'c' , the velocity of light in vacuum. This however does not contradict the special theory of relativity because a signal carried by waves moves not with phase velocity but with the group velocity, which is always less than 'c' as pointed out from equation (5.31) and (5.33). ($V_p.V_g = c^2$, so if $V_p > c$, obviously $V_g < c$)

Equation (5.33) shows that phase velocity is a function of wavelength even in free space. So unlike light waves, the de Broglie waves would show dispersion even in vacuum. This is the difference between de Broglie waves and light waves.